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Modifications of Maxwell's Equations Invariant under Galilean Transformations

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Abstract — In the special relativity, Maxwell's equations and the Lorentz force law are invariant under the Lorentz transformation. In this investigation, based on the modified Lorentz force law and the local-ether model of wave propagation recently developed, the Galilean-invariant counterparts of the wave equations of potential, the continuity equation, and of the Lorentz gauge are derived. Thereby, we derive the divergence and the curl relations of electric and magnetic fields. These relations have a unique feature of being Galilean invariant and present modifications of Maxwell's equations.

I. Introduction

It is generally expected from intuition that the electromagnetic force exerted on a charged particle should be invariant as observed in different inertial frames. In terms of electric and magnetic fields, the electromagnetic force exerted on a particle of charge q and velocity $\mathbf{v}$ is given by the famous Lorentz force law as $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. By resorting to the Lorentz transformation of space and time, it is known that the two fields together with the velocity transform in such a way that the Lorentz force is exactly identical to that given by the transformation of the time rate of change of kinematic momentum [1]. That is, the Lorentz force law is Lorentz invariant. Furthermore, the wave equations of potential, the continuity equation, and the Lorentz gauge, which are fundamental equations in electromagnetics, can be shown to be Lorentz invariant. Then, Maxwell's equations can be shown to be invariant under the Lorentz transformation [1].

Recently, we have presented a Galilean-invariant model of electromagnetic force law by proposing the augmented scalar potential, which is a modification of the electric scalar potential by incorporating a part associated with the velocities of the effector and the sources [2]. The augmented potential and the proposed force are Galilean invariant, since all the position vectors, time derivatives, velocities, and current density involved are Galilean invariant. Under the common *low-speed condition* where the sources drift very slowly in a matrix, the electromagnetic force can be given in terms of electric and magnetic fields in a form similar to the Lorentz force law. However, one fundamental difference is that the fields are defined explicitly in terms of potentials in a Galilean way. Thus, the fields as well as the electromagnetic force are invariant under Galilean transformations. In this investigation, based on the *local-ether model* of wave propagation [3], the Galilean-invariant counterparts of the wave equations of potential, the continuity equation, and the Lorentz gauge are derived. Thereby, the divergence and the curl relations of the Galilean-invariant fields are presented and then their relationship with Maxwell's equations are explored.

II. Modified Lorentz Force Law

Consider the electromagnetic force exerted on an effector of charge q due to an ensemble of source particles of charge density ρ_v drifting in a matrix of charge density ρ_m. The matrix can be a metal wire or a dielectric body and tends to electrically neutralize the mobile source particles. Under the common low-speed condition where the sources drift very slowly with respect to the matrix, the Galilean-invariant force model proposed in [2] has been shown to reduce to the modified Lorentz force law as

$$\mathbf{F}(\mathbf{r}, t) = q \left\{ \mathbf{E}(\mathbf{r}, t) + \mathbf{v}_{em} \times \mathbf{B}(\mathbf{r}, t) \right\}, \tag{1}$$

where $\mathbf{v}_{em} = \mathbf{v}_e - \mathbf{v}_m$ is the effector velocity with respect to the matrix and $\mathbf{v}_e$ and $\mathbf{v}_m$ are respectively the velocities of the effector and the matrix. The electric and magnetic fields in turn are defined explicitly in terms of potentials Φ and $\mathbf{A}$ as

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \left(\frac{\partial}{\partial t} \mathbf{A}(\mathbf{r}, t) \right)_m \tag{2}$$

and

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t), \tag{3}$$

where the time derivative $(\partial/\partial t)_m$ is referred specifically to the matrix frame with respect to which the matrix is stationary. The electric scalar and the magnetic vector potentials Φ and $\mathbf{A}$ in turn are defined based on the augmented potentials as

$$\Phi(\mathbf{r}, t) = \frac{1}{\epsilon_0} \int \frac{\rho_n(\mathbf{r}', t - R/c)}{4\pi R} dv' \tag{4}$$

and

$$\mathbf{A}(\mathbf{r},t)=\frac{1}{\epsilon_0 c^2}\int\frac{\mathbf{J}_n(\mathbf{r}',t-R/c)}{4\pi R}dv', \tag{5}$$

where the propagation range $R = |\mathbf{r}-\mathbf{r}'|$, the net charge density $\rho_n = \rho_v + \rho_m$, the neutralized current density $\mathbf{J}_n = \mathbf{v}_{sm}\rho_v$, $\mathbf{v}_{sm} = \mathbf{v}_s - \mathbf{v}_m$ is the drift velocity of the mobile sources with respect to the matrix, and $\mathbf{v}_s$ is the velocity of the source particles. It is of essence to note that the source velocity $\mathbf{v}_{sm}$ involved in the neutralized current density is referred to the matrix frame and hence this current density is Galilean invariant. Then, if the propagation time R/c in the potentials are neglected for the moment, potentials Φ and $\mathbf{A}$ are Galilean invariant. Further, it is noted that the time derivative in the definition of field $\mathbf{E}$ and the effector velocity in the force law (1) are referred to the matrix frame and hence are also Galilean invariant. Consequently, fields $\mathbf{E}$ and $\mathbf{B}$ defined in (2) and (3) in terms of potentials and hence the modified force law are all Galilean invariant.

III. Wave Equations of Potential Based on Local-Ether Model

Recently, we have presented a Galilean model of wave propagation whereby electromagnetic wave is supposed to propagate via a medium like the ether [3]. However, the ether is not universal. It is proposed that the region under sufficient influence of the gravity due to the Earth, the Sun, or another celestial body forms a local ether which in turn moves with the gravitational potential of the respective body. Thus, as well as earth's gravitational potential, the earth local ether is stationary in an ECI (earth-centered inertial) frame. Consequently, the earthbound propagation depends on earth's rotation but is entirely independent of earth's orbital motion around the Sun or whatever. This local-ether model has been adopted to account for a variety of propagation phenomena, particularly the GPS (global positioning system) Sagnac correction and the interplanetary radar echo time. Moreover, the local-ether model is still in accord with the Michelson-Morley experiment which is known to make the classical ether notion obsolete.

According to the local-ether model, it is supposed here that each individual source particle continuously excites the electric scalar potential and hence other related potentials, which in turn propagate radially outward from the emission position at the speed c with respect to the local-ether frame, independent of the motion of the source and the effector. That is, the position vectors $\mathbf{r}$ and $\mathbf{r}'$ and hence the propagation range R in potentials Φ and $\mathbf{A}$ defined in (4) and (5) are referred to the local-ether frame and R/c represents the propagation time from the source point $\mathbf{r}'$ at the instant $t-R/c$ of wave emission to the field point $\mathbf{r}$ at the instant t. Since the position vectors and the wave propagation are referred to a specific reference frame, potentials Φ and $\mathbf{A}$ are actually Galilean invariant, as they are for quasi-static case where propagation delay time can be neglected. Thereupon, the electromagnetic force law and fields $\mathbf{E}$ and $\mathbf{B}$ given in (1)-(3) are actually Galilean invariant.

By applying the Laplacian to both sides of the integral formulas of potential (4) and (5) and then expanding the Laplacians of the time-dependent charge and current densities divided by R [4], it can be shown that the wave equations for these Galilean-invariant potentials are given as

$$\nabla^2\Phi(\mathbf{r},t)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi(\mathbf{r},t)=-\frac{1}{\epsilon_0}\rho_n(\mathbf{r},t) \tag{6}$$

and

$$\nabla^2\mathbf{A}(\mathbf{r},t)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}(\mathbf{r},t)=-\mu_0\mathbf{J}_n(\mathbf{r},t), \tag{7}$$

where $\mu_0 = 1/\epsilon_0 c^2$ and the position vector $\mathbf{r}$ and the time derivative $\partial/\partial t$ are referred to the local-ether frame, the reference frame of the wave propagation.

Recall that in the neutralized current density $\mathbf{J}_n$, the drift velocity $\mathbf{v}_{sm}$ of mobile source particles of charge density ρ_v is referred to the matrix frame. Consequently, the conservation of charge leads to a relation between $\mathbf{J}_n$ and ρ_v as

$$\nabla\cdot\mathbf{J}_n(\mathbf{r},t)=-\left(\frac{\partial}{\partial t}\rho_v(\mathbf{r},t)\right)_m. \tag{8}$$

This relation is just the continuity equation, except that the time derivative applied to the charge density is referred specifically to the matrix frame. Furthermore, the matrix-frame continuity equation can be given in terms of the net charge density as

$$\nabla\cdot\mathbf{J}_n(\mathbf{r},t)=-\left(\frac{\partial}{\partial t}\rho_n(\mathbf{r},t)\right)_m, \tag{9}$$

since the matrix charge density ρ_m is time-independent in the matrix frame. After applying the divergence and the matrix-frame time derivative to both sides of the potential wave equations (7) and (6), respectively, it can be shown by using (9) and an argument given in [4] that the two potentials are related to each other in a form similar to the continuity equation as

$$\nabla \cdot \mathbf{A}(\mathbf{r},t) = -\frac{1}{c^2}\left(\frac{\partial}{\partial t}\Phi(\mathbf{r},t)\right)_m . \tag{10}$$

This potential continuity equation looks like the Lorentz gauge. However, the fundamental difference is that the time derivative applied to the scalar potential is referred specifically to the matrix frame.

IV. Modifications of Maxwell's Equations

In this section, we use the definitions of fields in terms of potentials and the potential wave equations to derive the divergence and the curl relations of electric and magnetic fields. By applying the curl and the divergence to both sides of the definitions of fields (2) and (3), respectively, one immediately has

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\left(\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)\right)_m \tag{11}$$

and

$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0. \tag{12}$$

Furthermore, by applying the divergence and the curl to both sides of the definitions of fields, respectively, one has

$$\nabla \cdot \mathbf{E}(\mathbf{r},t) = -\nabla^2\Phi(\mathbf{r},t) - \left(\frac{\partial}{\partial t}\nabla\cdot\mathbf{A}(\mathbf{r},t)\right)_m \tag{13}$$

and

$$\nabla \times \mathbf{B}(\mathbf{r},t) = -\nabla^2\mathbf{A}(\mathbf{r},t) + \nabla\nabla\cdot\mathbf{A}(\mathbf{r},t). \tag{14}$$

It is seen that these two relations involve the Laplacians of potential. By using the potential wave equations, the potential continuity equation, and the definition of electric field, the preceding two equations become

$$\nabla \cdot \mathbf{E}(\mathbf{r},t) = \frac{1}{\epsilon_0}\rho_n(\mathbf{r},t) + \frac{1}{c^2}\left\{\left(\frac{\partial^2\Phi}{\partial t^2}\right)_m - \frac{\partial^2\Phi}{\partial t^2}\right\} \tag{15}$$

and

$$\nabla \times \mathbf{B}(\mathbf{r},t) = \mu_0\mathbf{J}_n(\mathbf{r},t) + \frac{1}{c^2}\left(\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t)\right)_m + \frac{1}{c^2}\left\{\left(\frac{\partial^2\mathbf{A}}{\partial t^2}\right)_m - \frac{\partial^2\mathbf{A}}{\partial t^2}\right\}. \tag{16}$$

It is seen that these two relations then involve the differences between the second-order time derivatives of potentials referred to the matrix and the local-ether frames.

For quasi-static case or for the case where the matrix is stationary in the local-ether frame, these differences between derivatives vanish. Consequently, we arrive at

$$\left\{\begin{array}{rcll} \nabla \cdot \mathbf{E}(\mathbf{r},t) &=& \frac{1}{\epsilon_0}\rho_n(\mathbf{r},t) & \text{(17a)} \\ \nabla \times \mathbf{E}(\mathbf{r},t) &=& -\left(\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)\right)_m & \text{(17b)} \\ \nabla \cdot \mathbf{B}(\mathbf{r},t) &=& 0. & \text{(17c)} \\ \nabla \times \mathbf{B}(\mathbf{r},t) &=& \mu_0\mathbf{J}_n(\mathbf{r},t) + \frac{1}{c^2}\left(\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t)\right)_m . & \text{(17d)} \end{array}\right.$$

This set of relations presents modifications of Maxwell's equations, aside from the extra terms due to the differences between derivatives of potentials for general cases. The fundamental modifications are that the time derivatives in the two curl relations and the neutralized current density are all referred specifically to the matrix frame and that the position vector is referred to the local-ether frame. Moreover, the fields themselves are Galilean invariant. Thus, all of these divergence and curl relations of fields are Galilean invariant.

It can be shown that the extra term in (15) or (16) is much weaker than the other terms by a factor of the first order of normalized matrix speed v_m/c. For terrestrial phenomena, the local ether is stationary in an ECI frame and the speed due to earth's rotation is only about a millionth of the speed of light. Thereby, for the ordinary case where the matrix is moving at a sufficiently low speed with respect to the local-ether frame such that $v_m/c \ll 1$, the simplified modifications of Maxwell's equations (17) hold. Remark that the modifications of Maxwell's equations are based on the modified Lorentz force law (1), which in turn is valid under the low-speed condition. Thus, the simplified modifications of Maxwell's equations (17) are valid to the first order of normalized speed under the low-speed condition: $v_{sm} \lll c$

and $v_e, v_m \ll c$. This ordinary condition states that the effector, sources, and matrix move somewhat slowly with respect to the local-ether frame and the sources drift very slowly with respect to the matrix frame. Nevertheless, in the wave equations of field derived in the following section, these extra terms associated with the potential derivatives are taken into consideration.

V. Wave Equations of Field

From the two curl relations (11) and (16), the local-ether wave equation of electric and magnetic fields can be derived as

$$\nabla^2 \mathbf{E}(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{r},t) = \frac{1}{\epsilon_0}\nabla\rho_n(\mathbf{r},t) + \mu_0\left(\frac{\partial}{\partial t}\mathbf{J}_n(\mathbf{r},t)\right)_m \tag{18}$$

and

$$\nabla^2 \mathbf{B}(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{B}(\mathbf{r},t) = -\mu_0\nabla\times\mathbf{J}_n(\mathbf{r},t), \tag{19}$$

where the divergence relations and the definitions of electric and magnetic fields have been made use of. Alternatively, the wave equations of field can be derived by manipulating the wave equations of potential according to the definitions of fields. It is seen that the wave equation of electric field involves two time derivatives referred to different frames. In free space having no sources, the wave equations become a simpler form as

$$\nabla^2 \Psi(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi(\mathbf{r},t) = 0, \tag{20}$$

where Ψ denotes the scalar potential Φ or any Cartesian component of the vector potential $\mathbf{A}$, electric field $\mathbf{E}$, or magnetic field $\mathbf{B}$. It is seen that both electric and magnetic fields as well as the potentials propagate at the speed c with respect to the local-ether frame.

VI. Conclusion

Based on the Galilean-invariant electromagnetic force law under the ordinary low-speed condition, the force exerted on a charged particle can be given in terms of electric and magnetic fields. The fields in turn are given explicitly in terms of the electric scalar and the magnetic vector potentials. Further, based on the local-ether model of wave propagation, the propagation of the potentials is referred to the local-ether frame. Then, the potential wave equations and the potential continuity equation are derived. The position vectors, time derivatives, velocities, current density, and potentials involved are all Galilean invariant. Consequently, electric and magnetic fields are also Galilean invariant.

Thereafter, the divergence and the curl relations of electric and magnetic fields are derived from the definitions of fields in terms of potentials, the potential wave equations, and the potential continuity equation. It is found that for quasi-static case or the case where the matrix is stationary with respect to the local ether, these divergence and curl relations of fields look like Maxwell's equations. However, the fundamental modification is that the neutralized current density and the time derivatives in the curl relations are all referred specifically to the matrix frame. Therefore, all of these divergence and curl relations are Galilean invariant. Moreover, for general cases, the divergence relation of electric field and the curl relation of magnetic field are further modified with extra terms associated with the differences between derivatives of potentials in the matrix and the local-ether frames. These present modifications of Maxwell's equations. These extra terms are of the first order of normalized matrix speed with respect to the local-ether frame. For terrestrial phenomena, the local ether is stationary in an ECI frame and the speed due to earth's rotation is only about a millionth of the speed of light. Thus, the simplified modifications of Maxwell's equations are valid to the first order under the ordinary low-speed condition where the effector, sources, and matrix move slowly with respect to the local-ether frame and the sources drift very slowly with respect to the matrix frame.

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