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CONNECTIONS BETWEEN THE RIEMANN AND THE LORENTZ FORCE LAWS

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It is known that for the magnetic force due to a closed circuit, the Weber force law can be identical to the Lorentz force law. In this investigation it is shown that for both the electric and the magnetic force of the quasi-static case, the Riemann force law can be identical to the Lorentz force law, while the former is based on a potential energy depending on a relative speed and is in accord with Newton's law of action and reaction.

Key words: Lorentz force law, Riemann force law, potential energy, Newton's law of action and reaction

1. INTRODUCTION

The Lorentz force law together with Maxwell's equations forms the fundamental equations adopted by Lorentz in the development of electromagnetics. In addition to the Lorentz force law, the Weber and the Riemann force law were once adopted in the early development. A common feature of these three force laws is that they can be derived from Lagrange's equation by adopting a velocity-dependent potential energy. However, the latter two are almost obsolete in the literature.

It is known that for the magnetic force between two current-carrying wires, the Weber force law leads to the Ampère force law which in turn can be identical to the Biot-Savart (Grassmann) force law incorporated in the Lorentz force law. This identity holds for the ordinary situation where the force is due to a uniform current on a closed wire and the electrons forming the current drift at a low speed with respect to the wire. Further, under this ordinary situation it is shown in this investigation that for the electric and magnetic forces exerted on a charged particle of the quasi-static case, the Riemann force law is identical to the Lorentz force law. This identity, which seems not yet to be reported, is expected to be of significance since the Riemann force law is based on a potential energy depending on a relative speed and is in accord with Newton's law of action and reaction.

2. LORENTZ FORCE LAW AND ITS DERIVATION

It is well known that in the presence of electric and magnetic fields, the electromagnetic force exerted on a particle of charge q and velocity $\mathbf{v}$ is given by the Lorentz force law

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right). \tag{1}$$

The Lorentz force law can be given directly in terms of the scalar and the vector potential originating from the charge and the current density, respectively. That is,

$$\mathbf{F} = q\left(-\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times \nabla \times \mathbf{A}\right), \tag{2}$$

where Φ is the electric scalar potential and $\mathbf{A}$ is the magnetic vector potential. The term associated with the gradient of the scalar potential, with the time derivative of the vector potential, and the one with the particle velocity are known as the electrostatic force, the electric induction force, and the magnetic force, respectively. Quantitatively, the scalar and the vector potential are given explicitly in terms of the charge density ρ and the current density $\mathbf{J}$, respectively, by the volume integrals

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t)}{R} dv', \tag{3}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t)}{R} dv', \tag{4}$$

where $\mu_0\epsilon_0 = 1/c^2$, $R = |\mathbf{r} - \mathbf{r}'|$, and the time retardation R/c from the source point $\mathbf{r}'$ to the field point $\mathbf{r}$ is neglected for simplicity as the cases considered are quasi-static. It is noted that compared to the electrostatic force due to the scalar potential, both the electric induction force and the magnetic force due to the vector potential are of the second order of normalized speed with respect to c.

In classical mechanics, the force exerted on a particle of velocity $\mathbf{v}$ can be given by Lagrange's equation in conjunction with a term

U which is associated with the potential energy and depends on the particle velocity. That is,

$$\mathbf{F} = -\nabla U + \sum_i \hat{\boldsymbol{\imath}} \frac{d}{dt}\left(\frac{\partial U}{\partial v_i}\right), \tag{5}$$

where $v_i = \mathbf{v}\cdot\hat{\imath}$, $\hat{\imath}$ is a unit vector, and the index $i = x, y, z$.

It is known that the Lorentz force law (2) can be derived from Lagrange's equation by adopting the velocity-dependent potential energy U which in turn incorporates the scalar potential Φ and the vector potential $\mathbf{A}$. That is,

$$U = q\Phi - q\mathbf{v}\cdot\mathbf{A}. \tag{6}$$

This approach was pioneered by Clausius in 1877 [1, 2]. In the derivation the expansion $d\mathbf{A}/dt = \partial\mathbf{A}/\partial t + (\mathbf{v}\cdot\nabla)\mathbf{A}$ and the identity $\nabla(\mathbf{v}\cdot\mathbf{A}) - (\mathbf{v}\cdot\nabla)\mathbf{A} = \mathbf{v}\times\nabla\times\mathbf{A}$ have been used. It is seen that the electric induction force is similar to the magnetic induction force in their physical origin, where the latter is associated with the term $(\mathbf{v}\cdot\nabla)\mathbf{A}$ and is an ingredient of the magnetic force. In the preceding potential energy U, the velocity $\mathbf{v}$ and the velocity of the mobile charged particles involved in the potential $\mathbf{A}$ are not relative. Thus the potential energy and hence the derived force are not frame-invariant under Galilean transformations. Furthermore, the derived force between two moving charged particles is not in accord with Newton's law of action and reaction. On the other hand, it is known that the Lorentz force law is invariant under the Lorentz transformation.

Consider the case where the force exerted on a wire segment carrying a current is due to another current element. Suppose the current elements are electrically neutralized, the two wires are stationary with respect to each other, and the currents are static. Thus the electrostatic and electric induction forces vanish and only the magnetic force is observable. Quantitatively, for the magnetic force exerted on a wire segment of directed length $d\mathbf{l}_1$ and carrying a current I_1 due to another current element $I_2 d\mathbf{l}_2$, the Lorentz force law reduces to the Biot-Savart (Grassmann) force law

$$\mathbf{F} = -\frac{\mu_0}{4\pi} I_1 I_2 \frac{1}{R^2}\left[\hat{\mathbf{R}}(d\mathbf{l}_1\cdot d\mathbf{l}_2) - (d\mathbf{l}_1\cdot\hat{\mathbf{R}})d\mathbf{l}_2\right], \tag{7}$$

where $\hat{\mathbf{R}}$ is a unit vector pointing from element 2 to element 1 and R is the separation distance between them.

3. WEBER FORCE LAW AND AMPÈRE FORCE LAW

In as early as 1846, Weber presented a second-order generalization of Coulomb's law for electrostatic force. The Weber force law can be derived from a velocity-dependent term associated with the potential energy, which, for the force exerted on a particle of charge q_1 and velocity $\mathbf{v}_1$ due to another particle of charge q_2 and velocity $\mathbf{v}_2$, is given by [1–3]

$$U = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{R} \left(1 + \frac{u_{12}^2}{2c^2}\right), \tag{8}$$

where R is the relative distance between the two charged particles, $u_{12} = (\mathbf{v}_1 - \mathbf{v}_2)\cdot\hat{\mathbf{R}}$ is the radial relative speed between them, and $\hat{\mathbf{R}}$ points from particle 2 to particle 1.

As the potential energy depends on the radial speed, it is of convenience to use the chain rule to express Eq. (5) in the form

$$\mathbf{F} = -\nabla U + \sum_i \hat{\boldsymbol{\imath}} \frac{d}{dt} \left(\hat{\boldsymbol{\imath}}\cdot\hat{\mathbf{R}} \frac{\partial U}{\partial u_1}\right), \tag{9}$$

where v_i in (5) is understood as v_{1i} $(= \mathbf{v}_1\cdot\hat{\imath})$. Then, by using the identity $\nabla u_{12} = d\hat{\mathbf{R}}/dt = (\mathbf{v}_{12} - u_{12}\hat{\mathbf{R}})/R$, the preceding force formula becomes the form given in [2]

$$\mathbf{F} = \hat{\mathbf{R}} \frac{1}{R} U + \hat{\mathbf{R}} \frac{d}{dt} \frac{\partial U}{\partial u_1}. \tag{10}$$

In dealing with the time derivative associated with the potential energy, one uses the expansion $d(u_{12}/R)/dt = (du_{12}/dt)/R - u_{12}^2/R^2$, as both of the variations of u_{12} and R contribute to the time derivative. Further, by expanding the derivative du_{12}/dt, one has the Weber force law [1–3]

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{\hat{\mathbf{R}}}{R^2} \left(1 + \frac{v_{12}^2}{c^2} - \frac{3}{2} \frac{u_{12}^2}{c^2} + \frac{\mathbf{R} \cdot \mathbf{a}_{12}}{c^2}\right), \tag{11}$$

where $\mathbf{a}_{12}$ denotes the relative acceleration. It is noted that the force is always along the radial direction represented by $\hat{\mathbf{R}}$. As the involved distance, velocity, and acceleration are all relative between the two particles, the Weber force is frame-invariant simply under Galilean transformations and is in accord with Newton's law of action and reaction.

Consider the case where the magnetic force is due to a neutralized current where the mobile charged particles forming the current are actually embedded in a matrix, such as electrons in a metal wire. The ions that constitute the matrix tend to electrically neutralize the mobile particles. Suppose that the various ions and hence the neutralizing matrix move at a fixed velocity $\mathbf{v}_m$. Thus the mobile charged particles drift at the speed v_{2m} relative to the matrix. Ordinarily, the drift speed v_{2m} is quite low due to the collision of electrons against ions. Thus, based on the Weber force law, the force due to a neutralized current element exerted on a charged particle of relative velocity $\mathbf{v}_{1m}$ can be given by superposing the forces due to the electron and ion. Thus one has the force law between the current element and the particle

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\epsilon_0 c^2} \frac{\hat{\mathbf{R}}}{R^2} \left(-2\mathbf{v}_{1m} \cdot \mathbf{v}_{2m} + 3u_{1m}u_{2m} - \mathbf{R} \cdot \mathbf{a}_{2m}\right), \tag{12}$$

where it has been supposed that the drift speed v_{2m} is sufficiently low as it is ordinarily and thus those terms associated with the second order of $\mathbf{v}_{2m}$ are neglected.

Further, consider two neutralized current elements flowing on two wire segments which in turn are stationary with respect to each other. Then, by superposing the forces exerted on the electron and ion, one has the force law between the two current elements

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\epsilon_0 c^2} \frac{\hat{\mathbf{R}}}{R^2} \left(-2\mathbf{v}_{1m} \cdot \mathbf{v}_{2m} + 3u_{1m}u_{2m}\right). \tag{13}$$

Since $\mathbf{v}_{1m}$ and $\mathbf{v}_{2m}$ are relative, this force is Galilean invariant. And as these velocities appear in a symmetric way, the action of a current element on itself then cancels out. As $q_1\mathbf{v}_{1m}$ and $q_2\mathbf{v}_{2m}$ correspond to $I_1 d\mathbf{l}_1$ and $I_2 d\mathbf{l}_2$, respectively, the preceding formula becomes the Ampère force law

$$\mathbf{F} = -\frac{\mu_0}{4\pi} I_1 I_2 \frac{\hat{\mathbf{R}}}{R^2} \left[2(d\mathbf{l}_1 \cdot d\mathbf{l}_2) - 3(d\mathbf{l}_1 \cdot \hat{\mathbf{R}})(d\mathbf{l}_2 \cdot \hat{\mathbf{R}})\right]. \tag{14}$$

It can be shown that the magnetic force predicted from the Ampère law is identical to the one from the Biot-Savart law (7), when the force is due to a closed circuit with uniform current as it is ordinarily. Such an identity has been proved by two elegant but similar approaches by using vector identities [4, 5], where the magnetostatic condition, under which the divergence of the current density is zero, is assumed. A closed circuit with uniform current is a common case of this condition.

4. RIEMANN FORCE LAW

The electromagnetic force law can be derived alternatively from a potential-energy term incorporating the relative speed, instead of the radial relative speed. That is, [1, 3]

$$U = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{R} \left(1 + \frac{v_{12}^2}{2c^2}\right). \tag{15}$$

This velocity-dependent potential energy was introduced by Riemann in 1861 [2] and is almost ignored at the present time. Then Lagrange's equation immediately leads to the Riemann force law [1, 3]

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\epsilon_0}\left[\frac{\hat{\mathbf{R}}}{R^2}\left(1+\frac{v_{12}^2}{2c^2}\right) - \frac{1}{c^2R^2}u_{12}\mathbf{v}_{12} + \frac{1}{c^2R}\mathbf{a}_{12}\right], \tag{16}$$

where, as in deriving (11), one uses the expansion

$$\frac{d}{dt}\frac{\mathbf{v}_{12}}{R} = \frac{\mathbf{a}_{12}}{R} - \frac{u_{12}\mathbf{v}_{12}}{R^2}, \tag{17}$$

as both of the variations of $\mathbf{v}_{12}$ and R contribute to the time derivative. Physically, the derivative $d(\mathbf{v}_{12}/R)/dt$ is associated with the time rate of change in the potential energy actually experienced by the affected particle. And the term with $u_{12}\mathbf{v}_{12}$ in the preceding force formula is associated with the variation of the experienced potential energy due to the relative displacement between the affected and the source particle. It is of essence to note that the potential energy and the force depend on the relative velocity and distance and hence they are independent of the choice of reference frames in uniform motion of translation. Furthermore, the Riemann force law as well as the Weber force law is in accord with Newton's third law of motion.

Now we consider the ordinary case where the force is due to a neutralized current element with a sufficiently low drift speed v_{2m}. By superposition the Riemann force exerted on a charged particle moving at a velocity $\mathbf{v}_{1m}$ relative to the matrix is then given by

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\epsilon_0 c^2}\left[\frac{1}{R^2}(-\hat{\mathbf{R}}\mathbf{v}_{1m}\cdot\mathbf{v}_{2m} + u_{1m}\mathbf{v}_{2m} + u_{2m}\mathbf{v}_{1m}) - \frac{1}{R}\mathbf{a}_{2m}\right]. \tag{18}$$

Omitting the acceleration term, a similar force formula between two current elements can be found in [2, 3]. When the current-carrying wire forms a loop C_2 over which the current is uniform and thus the neutralization remains, the force becomes

$$\mathbf{F} = \frac{q_1}{4\pi\epsilon_0 c^2}\oint_{C_2}\frac{\rho_l}{R^2}(-\hat{\mathbf{R}}\mathbf{v}_{1m}\cdot\mathbf{v}_{2m} + u_{1m}\mathbf{v}_{2m})dl \;-\; q_1\frac{\partial\mathbf{A}}{\partial t}, \tag{19}$$

where ρ_l denotes the line charge density of the mobile particles of the neutralized loop, the vector potential is given by

$$\mathbf{A} = \frac{\mu_0}{4\pi}\oint_{C_2}\frac{\rho_l\mathbf{v}_{2m}}{R}dl, \tag{20}$$

and we have made use of the consequence that a uniform current $(\rho_l v_{2m})$ leads to

$$\oint_{C_2}\frac{\rho_l u_{2m}}{R^2}dl = 0. \tag{21}$$

Similarly, for a volume current density under the magnetostatic condition, it can be shown that the contribution corresponding to that of the term $\rho_l u_{2m}$ cancels out collectively. It is noted that the time derivative $\partial \mathbf{A}/\partial t$ associated with the electric induction force is actually referred to the matrix frame (in which the matrix is stationary) so that the variation of $\mathbf{v}_{2m}$ contributes to this derivative, while the variation of R does not as its effect has been counted in the term with $u_{1m}\mathbf{v}_{2m}$ in (19).

Further, by using vector identities, the force given by (19) can be written as

$$\mathbf{F} = q_1 \left[\mathbf{v}_{1m} \times (\nabla \times \mathbf{A}) - \frac{\partial \mathbf{A}}{\partial t} \right]. \tag{22}$$

It is essential to note that the preceding formula looks like the Lorentz force law under neutralization. However, the current density generating the potential $\mathbf{A}$, the time derivative of $\mathbf{A}$ in the electric induction force, and the particle velocity connecting to $\nabla \times \mathbf{A}$ in the magnetic force are all referred specifically to the matrix frame. It is of significance to note that this specific frame has been adopted tacitly in common practice with the magnetic and induction forces, such as in dealing with the magnetic deflection in a cathode-ray tube or mass spectrometer, with the magnetic force in a motor, and with the electric induction force in an inductor or transformer. Thus, for the ordinary case where the electric and magnetic forces are due to closed circuits with uniform current and low drift speed, the Riemann force law is actually identical to the Lorentz force law, though the former is Galilean invariant and in accord with Newton's law of action and reaction.

Recently, based on a wave equation a time evolution equation similar to Schrödinger's equation is derived. From the evolution equation an electromagnetic force given in a form quite similar to Lagrange's equation in conjunction with the potential energy (15) is derived [6, 7]. Thus a quantum-mechanical basis for the Riemann force law has been provided. Further, the divergence and the curl relations for the corresponding electric and magnetic fields are derived. Apart from some minute terms, these four relationships are just Maxwell's equations, with the exception that the velocity determining the involved current density is also relative to the matrix [7].

5. CONCLUSION

The Weber and the Riemann force laws are derived from Lagrange's equation in conjunction with a potential energy depending on the radial relative speed and on the relative speed, respectively. As the involved speeds are relative, these two force laws are frame-invariant simply under Galilean transformations and are in accord with Newton's law of action and reaction. Ordinarily, the magnetic force is due to uniform currents on closed wires. Under this situation, the Weber force law can lead to the Ampère force which in turn is identical to the magnetic force given by the Lorentz force law. Further, it is shown that under the magnetostatic condition with low drift speed, the Riemann force law is identical to the Lorentz force law for the electric force in addition to the magnetic force. However, the current density generating the vector potential, the time derivative of this potential in the electric induction force, and the particle velocity connecting to this potential in the magnetic force are all referred specifically to the matrix frame. The adoption of the matrix frame does not actually represent a departure from the standard electromagnetic theory, as it has been done tacitly in common practice with the Lorentz force law.

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